AFRL-SR-AR-TR-07-0392

## REPORT DOCUMENTATION PAGE



**1. REPORT DATE (DD-MM-YYYY)** 01 OCT 2007

**2. REPORT TYPE** FINAL REPORT

**3. DATES COVERED (From - To)** 1 APR 03 - 30 SEP 06

**4. TITLE AND SUBTITLE**
COMPUTATIONAL METHODS FOR DESIGN, CONTROL AND OPTOMIZATION

**5a. CONTRACT NUMBER**

**5b. GRANT NUMBER** F49620-03-1-0243

**5c. PROGRAM ELEMENT NUMBER**

**5d. PROJECT NUMBER**

**5e. TASK NUMBER**

**5f. WORK UNIT NUMBER**

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**8. PERFORMING ORGANIZATION REPORT NUMBER**

**9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)**
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SUITE 325, ROOM 3112
ARLINGTON, VA 2203-1768
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**10. SPONSOR/MONITOR'S ACRONYM(S)**

**11. SPONSOR/MONITOR'S REPORT NUMBER(S)**

**12. DISTRIBUTION/AVAILABILITY STATEMENT**
APPROVE FOR PUBLIC RELEASE, DISTRIBUTION UNLIMITED.

**13. SUPPLEMENTARY NOTES**

**14. ABSTRACT**
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**15. SUBJECT TERMS**

| 16. SECURITY CLASSIFICATION OF: | | | 17. LIMITATION OF ABSTRACT | 18. NUMBER OF PAGES | 19a. NAME OF RESPONSIBLE PERSON |
|---|---|---|---|---|---|
| a. REPORT | b. ABSTRACT | c. THIS PAGE | | | John Burns |
| | | | | | 19b. TELEPHONE NUMBER (Include area code) |



20071015187

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# COMPUTATIONAL METHODS FOR DESIGN, CONTROL AND OPTIMIZATION

## FINAL REPORT ON

AFOSR GRANT F49620-03-1-0243

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## ABSTRACT

This report contains a final report on the activities supported under the Air Force AFOSR Grant F49620-03-1-0243 during the period 1 April 2003 through 30 September 2006. The goal of the effort is to produce rigorous mathematical foundations and practical computational tools for design, control and optimization of hybrid systems governed by partial differential equations that are typical in aerospace systems. The focus of the research is on non-normal systems of the type that occurs when one linearizes around a non-trivial state or when the model describes an interconnected system such as a fluid-structure interaction. The approach is based on design-then-approximate methods to guide the construction of the efficient algorithms. This approach requires that one uses distributed parameter theory to characterize the optimal solution and then introduces approximations at the last stage of the analysis.

> **The general objective** of the research was the development of rigorous and practical numerical algorithms for optimization, design and control of hybrid physics based systems with applications to aerospace systems. In addition we developed a computational environment and research software tools that engineers can effectively use to design and optimize aerospace systems. The goal was to provide a theoretical framework for the rigorous analysis of design algorithms that combine numerical simulation codes, approximate sensitivity calculations and optimization codes.
>
> **The fundamental approach** was based on "design-then-approximate" methods. These methods introduce numerical approximations at the last stage and allow us to take advantage of the physics of the problem in order to develop efficient and accurate algorithms This approach has been shown to be extremely useful in other areas of flow control, optimization and aerodynamic design. approximate sensitivity calculations and optimization codes.

The effort was built on a highly integrated interdisciplinary research program with the objective of developing efficient computational algorithms for optimal aerodynamic design and control of fluid/structure systems described by time varying partial differential equations.

## Summary of Accomplishments

We accomplished several important goals which will be detailed below. However, during the project we:

- Produced more than 55 scientific papers and 1 book,
- Made more than 100 presentations at conferences and colloquium,
- Directed more than 8 graduate students and 1 postdoc,
- Worked with more than 16 visitors, representing 6 different countries,
- Continued to assisted AeroSoft in the development of AeroSoft's *SENSE* software package,
- Made several visits to AFRL at Wright-Patterson and Eglin to work with Air Force personnel.

## Detailed Accomplishments

This section contains a brief description of some research accomplishments and provides a brief indication of the significance and potential applications.

**Accomplishment 1: Optimal Actuator Placement with Spatially Distributed Disturbances.** We developed a number of new design objectives to be used for sensor/actuator placement. These extend the usual LQR-type control cost used in many optimal placement strategies by considering spatially distributed disturbance functions. For example, consider the standard optimal LQR-cost problem defined by the system

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}(\alpha)u(t), \quad z(0) = z_0 \in Z$$

with cost function

$$\mathcal{J}(u, \alpha) = \int_0^\infty [\langle \mathcal{Q}z(t), z(t)\rangle_Z + u(t)^T Ru(t)]dt,$$

where $\alpha$ is a "parameter" that defines the actuator location. A sensor location problem may be defined by finding value of $\alpha_*$ that minimizes the LQR cost over a set of possible initial data $\bar{Z}$. *Minimize*

$$\max_{z_0 \in \bar{Z}} J(u, z_0) = \max_{z_0 \in \bar{Z}} \langle \Pi(\alpha)z_0, z_0\rangle$$

over all admissible locations $\alpha$ (here $\Pi$ is the solution to the algebraic Riccati equation, ARE). This measure works well for a number of problems. However, it does not extend to the case of multiple actuators (actuator locations are chosen to be co-located). As a result, we are looking at control performance measures that incorporate robustness. We continue to seek appropriate quantification of the best actuator location.

- **New Findings:** This work is a preliminary step in the development of general computational tools for optimal placement of sensors and actuators. We found that the continuous sensitivity equation method provides an efficient means for computing the derivatives needed in the optimization loop.

- **Significance and Potential Applications:** The use of optimization based placement algorithms has the potential for considerable payoffs in aerospace applications. In particular, the need to use full unsteady of aerodynamic models to capture the motions of a MAV in flight leads to complex optimization problems with shape as a design variable. Earlier work on the sensitivity equation method can now be used to attack this problem once a good design/cost function has been defined.

**Accomplishment 2: Developed an Adaptive Finite Element Scheme for Computing Functional Gains.** By using adaptive finite element schemes we were able to produce low order approximations of Riccati equations that define functional feedback gains. In addition, these low order approximations naturally lead to low order controllers.

- **New Findings:** We established that the use of non-uniform meshes for functional gain computations can significantly improve accuracy and convergence of the numerical approximations. Adaptive meshing can produce accurate solutions to the Riccati equations that define the gains with a "small" number of degrees of freedom.

- **Significance and Potential Applications:** The results point to two important issues that should be addressed in developing an adaptive method based on adaptive mesh refinement. First, the method should adapt on the solutions to the Riccati equation so that the support of the functional gain "drives" the mesh refinement scheme. This will ensure that the gains are computed on their support and that the size of the approximating system is kept at a minimum. Finally, the use of such methods could be considered as a first step in the construction of practical low order feedback controllers for fluid flow problems. The improved robustness and computational efficiency of this approach makes it practical for a wide class of problems. The significance is that very low order controllers can be designed which closely resemble the performance of impractical full state feedback designs.

**Accomplishment 3: Constructed an Improved "Group Finite Element" Scheme for Computing Feedback Controllers for Boundary Layer Control.** This effort lead to several new approaches to feedback control where the control appears in a boundary layer.

- **New Findings:** We constructed a new computational scheme for boundary control of PDEs with highly sensitive boundary layers similar to typical channel flow problems. The functional gains that define the optimal controllers have near wall support and suggest that it may be possible to design practical low order wall controllers.

- **Significance and Potential Applications:** Although we were not able to provide a rigorous convergence proof, the numerical results indicate that the scheme converges rapidly. The problem of convergence for the full nonlinear problem needs to be addressed. In addition, if the scheme can be extended to external flows, then it has potential to be used in MAV and other fluid flow applications. The implications for active control of complicated fluid flows are enormous.

**Accomplishment 4: Developed New Approximate Deconvolution Boundary Conditions for Large Eddy Simulation.** For the foreseeable future, Large-eddy Simulation (LES) is the only practical approach to capturing detailed structures in complex flows. One of the main challenges for LES is specification of efficient, general boundary conditions for the filtered variables. There are essentially two ways to treat boundary conditions in LES. The first, known as Near Wall Resolution (NWR), is to decrease the filter width to zero at the boundaries. This requires fine meshes near walls. The second, referred to as Near Wall Modeling (NWM), employs a coarse discretization near boundaries, and is developed by using boundary layer theory. We developed new boundary conditions for LES based on *approximate deconvolution*. Note that current LES boundary conditions are not up to this task: NWR would lead to a prohibitively high computational cost and the needed time-dependent boundary layer theory for NWM is not available. Our new boundary conditions avoid these two roadblocks and they are efficient and general.

- **New Findings:** We introduced the ADBC algorithm, a new set of boundary conditions for LES. The ADBC algorithm is based on an approximate deconvolution approach. The new boundary conditions are computationally efficient and general (they function in cases where the boundary layer theory is not available). These two features make the ADBC algorithm well suited for turbulent flows with time-dependent boundary conditions, such as those in a closed-loop flow control setting.
- **Significance and Potential Applications:** An advantage of these Approximate Deconvolution Boundary Conditions (ADBC) is that they are suited for turbulent flows with time-dependent boundary conditions. One such application is flow control, where for example, blowing and suction on the surface of an airfoil can be used to reduce the skin-friction drag. These results have the potential for considerable payoffs in MAV applications. In particular, the need to use full unsteady of aerodynamic models to capture the motions of a MAV in flight leads to complex optimization problems with shape as a design variable. Earlier work on the sensitivity equation method can now be used to attack this problem once a good design/cost function has been defined.

**Accomplishment 5: Developed an Adaptive Finite Element Scheme for Computing Functional Gains and Estimating Condition Numbers** In order to generate a computational toolbox for active feedback flow control, it is clear that one must develop new computational schemes for solving Riccati, Lyapunov and other control equations described by partial differential equations. Also, new theories need to be developed to justify these schemes since the equations important for control design have not been considered by the classical CFD community. We have developed new methods based on combining adaptive finite elements, stabilized Petrov-Galerkin methods and Chandrasekhar type algorithms. In this effort we raised several basic issues that have, until now, been ignored in the standard "flow control" literature. In particular, we discovered that straightforward application of typical CFD methods can (and often do) lead to extremely ill-conditioned Riccati equations. In addition, we made the first steps in dealing with this ill-conditioning.

- **New Findings:** We demonstrated that the use of non-uniform meshes for functional gain computations can significantly improve accuracy and convergence of the numerical approximations. Adaptive meshing can produce accurate solutions to the Riccati equations that define the gains with a "small" number of degrees of freedom. However, we discovered that non-uniform meshes can lead to ill-conditioned finite dimensional Riccati equations. This observation lead to the investigation of conditioning for Riccati equations and possible pre-conditioners. We also discovered that ill-conditioning can occur when one uses "standard" finite element methods to approximate convection dominated flows. Our preliminary findings indicate that stabilization (SUPG) methods can be used to pre-condition the Riccati equation. This is the first step in dealing with this issue and since flow control problems are modelled by convection-diffusion equations, this ill-conditioning must be dealt with before practical flow control tools can be fully developed.

- **Significance and Potential Applications:** Adaptive and LES/reduced order methods will be major components of any practical flow control toolbox. These methods must produce accurate and robust solutions to Riccati equations so numerical conditioning is essential. In addition to payoff in flow control, this research will provide more accurate finite dimensional algorithms for a variety of large control systems. Therefore, the resulting computational tools will be also be applicable to large inflatable structures of considerable interest to the DoD and Air Force.

**Accomplishment 6: Developed the beginnings of a rigorous theory that can be used as a framework to understanding flow transition in shear flows and for designing feedback controllers to delay transition.** It has long been known that classical linear hydrodynamic stability theory fails to accurately predict transition to turbulence in shear flows. In particular, there is no single transition "scenario" that applies to channel flows (Poiseuille flows, Couette flow) and pipe flows. Over the past 75 years many complex "transition theories" have been put forward, but still none of these scenarios accurately predict the critical Reynolds number except in very special cases. During the past 10 years a new mathematical approach to hydrodynamic stability theory and transition has emerged. There are two basic ideas in this new approach. The ideas put forth by Trefethen, Henningson and their co-workers make use of pseudospectrum to predict large transient growths and transition is considered a problem of small stability margins. Farrell, Ioannou, Bamieh, Dahleh and their co-workers describe the transition as a response to a small disturbance (wall roughness, body force, etc.). This approach is based on robust control ideas. All these approaches are based on some linear hydrodynamic stability theory. Although both ideas have lead to a better understanding of transition and these methods have been used to more accurately predict critical Reynolds numbers, neither approach provides an understanding of what actually causes the transition. We provided the foundations for a rather general theory that combines classical ideas from infinite dimensional dynamical systems with the two new approaches above. This approach allows us to include (and understand) the impact of the non-linearities on the process of transition. We tested this new theory on model problems (including Burgers' equation) and it appears that we can provide a rigorous mathematical framework to provide at least a partial explanation of how transition might occur.

- **New Findings:** We developed a simple mathematical theory that provides a rigorous foundation for studying the process of transition. The idea is to combine robust control theory, sensitivity analysis and bifurcation theory for infinite dimensional theory to show that transition can occur due to high sensitivity and uncertainty in the system equations. In particular, the equations are "infinitely sensitive" and transition occurs because the system is "near" another dramatically different dynamical system. We have shown this to be true for Burgers' equation and other models that are typical in fluid flows. The idea is based on "bifurcation under uncertainty" and allows one to study the impact of unmodelled parameters.
- **Significance and Potential Applications:** In order to develop practical control design tools for feedback control of fluid flows, one needs to have an understanding (a rigorous theory) of the dynamical process being approximated. The theory we are building has the potential to provide a mathematical foundation that can be exploited in the development of computational tools. The implications for active control of complicated fluid flows are enormous. Moreover, if successful such a theory would provide a mathematical approach to the old problem of transition to turbulence.

The problem of controlling or delaying transition to turbulence in shear flows has been the subject of numerous papers over the past twenty years. Although there is no single mathematical framework that describes transition for all possible flows, new approaches to (non-classical) linear hydrodynamic stability theory have provided tremendous improvements in the fundamental understanding of this process. In particular, ideas from robust control theory have been used to develop new linear theories in an attempt to explain some of the failures of classical linear hydrodynamic stability theory. We have suggested a new scenario to explain the mechanism and to help design feedback controllers for transition control. The basic tool used to develop this scenario falls under the category of "bifurcation analysis under uncertainty" and matches many of the observed flows. In addition, we employed sensitivity analysis to deal with this uncertainty and to construct feedback controllers for transition control. We established that feedback controllers can delay transition and alter the global dynamics of such systems.
The mathematical model common to many flow control problems has the form

$$\dot{z}(t) = [\frac{1}{R}A_0 + \mathcal{R}]z(t) + F(z(t)) + Bu(t), \quad z(0) = z_0,$$

where $A_0$ is a self-adjoint operator, $R > 0$, $\mathcal{R}$ is a perturbation operator and $A = [\frac{1}{R}A_0 + \mathcal{R}]$ is non-normal. The nonlinear term $F(\cdot)$ is conservative. i.e. $\langle F(z), z\rangle = 0$.

The linear part of such non-normal systems is extremely important in understanding sensitivity and control design. However, it is the perturbation of the conservative nonlinear term that might explain a transition mechanism. Even a small perturbation in the boundary condition (wall roughness, forced vibration, etc.) can eliminate the conservation condition. In particular, if $\varepsilon \neq 0$ then the perturbed nonlinear term becomes

$$F_\varepsilon(z) = F(z) + G\varepsilon$$

so that the nonlinear term is no longer conservative nor definite. It is the perturbation of this conservative nonlinear term that can provide the transition mechanism and lead to a subcritical "bifurcation under uncertainty". Bifurcation diagrams such as shown in Figure 2 are indicative of this type of mechanism.

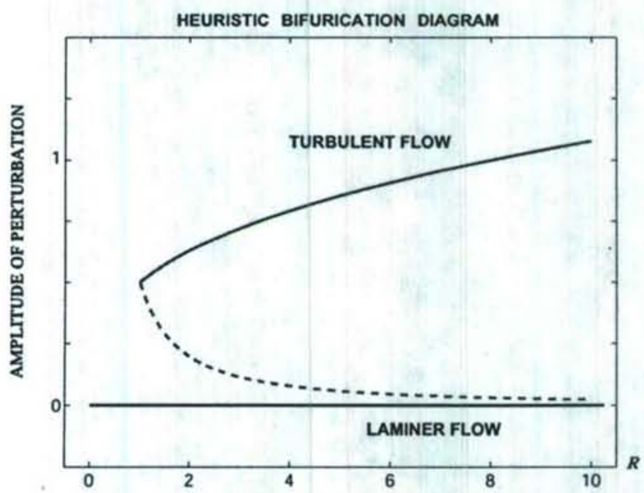


Figure 1: Heuristic bifurcation diagram for low dimensional models. The laminar flow is stable for all $R > 0$ but the stability radius decays to 0 as $R \longrightarrow +\infty$. An initial state must be above the dashed blue line to transition.

In addition to providing a framework to help with the fundamental understanding of certain transition, the abstract formulation above can be used to quantify sensitivity and uncertainty. Moreover, we combined feedback control theory with sensitivity analysis to develop a strategy for turning on the feedback controller. These results are reported in papers [7] and [8].

**Accomplishment 7: A New Sufficient Condition for Mesh Independence of Riccati Solvers.** The development of accurate numerical algorithms for solving the large scale Riccati equations that arise in feedback control of PDE systems requires special techniques. One of the most effective methods is the Kleinman-Newton algorithm and mesh independence was established for a large class of PDE systems.

- **New Findings:** We developed a mathematical theory that provides a rigorous foundation for studying the mesh independence of the Kleinman-Newton algorithm.

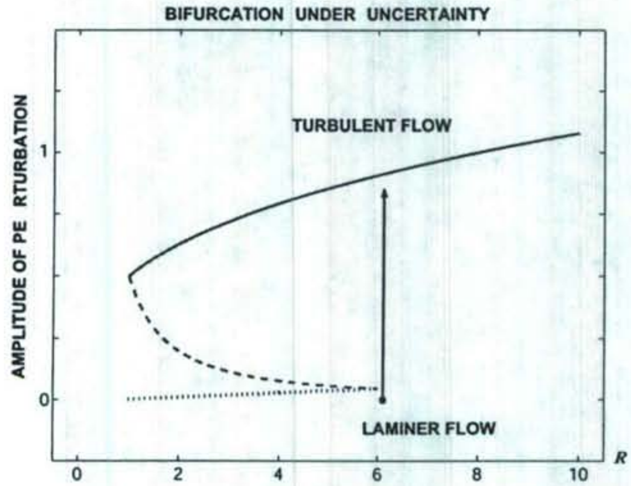


Figure 2: A heuristic bifurcation diagram under uncertainty. The small constant disturbance produces a non-conservative nonlinear term which leads to a subcritical bifurcation. The laminar flow state is no longer an equilibrium for $R > R_{crit}$ and transition occurs.

This theory required that new techniques be developed to prove convergence since the non-linear operator associated with the Riccati equation is **NOT** Frechet differentiable. Standard Newton type methods do not apply in this case. The important issues are understanding which properties of the control system are essential to the mesh independence principle. We showed that compactness of the controlled output operator is necessary for mesh independence. Also we provided convergence rates that depend on rates of dual convergence of the dual operators.

- **Significance and Potential Applications:** In order to develop practical computational tools for feedback control of large PDE type systems, one needs efficient algorithms. These results show that formulating the control problem so that the controlled output operator is compact is essential. Also, the specific approximation scheme must be dual convergent to obtain mesh independence. Therefore, for non-normal systems that occur in fluid/structure interactions one must be careful to generate such approximations.

Consider the linear quadratic regulator (LQR) problem in an infinite dimensional Hilbert space setting. Let $U$, $H$ and $Y$ be Hilbert spaces over the reals. If $Z$ and $W$ are any two Hilbert spaces, then we denote by $\mathcal{L}(Z,W)$ the linear space of linear bounded operators from $Z$ into $W$. In the special case where $W = Z$, we set $\mathcal{L}(Z) = \mathcal{L}(Z,Z)$. The system equation is given in state space form by

$$\dot{z}(t) = Az(t) + Bu(t), \quad t \geq 0, \quad z(0) = z_0 \in H, \tag{1}$$

where $A$ generates a strongly continuous semigroup on H and $B \in \mathcal{L}(U,H)$. Let $C \in \mathcal{L}(H,Y)$ and define the quadratic cost function $J(u)$ by

$$J(u) = \int_0^\infty \left(\|Cz(s)\|^2 + \|u(s)\|^2\right) ds, \tag{2}$$

The LQR control problem is to minimize the quadratic cost $J(u)$ over all controls $u \in L^2(0,\infty;U)$. It is well known that under certain assumptions the optimal control is given by state feedback $u_{opt} = -K_{opt}z(t)$ where

$$K = B^* X_{opt}, \tag{3}$$

$z(s)$ is the solution to (1) for a given control $u \in L^2(0,\infty;U)$.and $X_{opt} = X_{opt}^* \in \mathcal{L}(H)$ is a solution of an abstract algebraic Riccati (PDE) operator equation of the form

$$A^*X + XA - XBB^*X + C^*C = 0. \tag{4}$$

We focused on the problem of developing numerical schemes that yield convergent and mesh independent approximations of the infinite dimensional Riccati equation

$$\mathcal{F}(X) = A^*X + XA - XBB^*X + C^*C = 0. \tag{5}$$

Applying Newton's method to $\mathcal{F}(X) = 0$, leads to the iteration

$$(A - BB^*X_k)^*X_{k+1} + X_{k+1}(A - BB^*X_k) = -BB^*X_k - C^*C.$$

However, in order to compute one must introduce an approximation scheme. We considered a sequence of approximating problems defined by $(H^N, A^N, B^N, C^N)$, where $H^N \subset H$ is a sequence of finite dimensional subspaces of $H$ and $A^N \in \mathcal{L}(H^N, H^N)$, $B^N \in \mathcal{L}(U, H^N)$ and $C^N \in \mathcal{L}(H^N, Y)$ are bounded linear operators. Let $P^N : H \to H^N$ denote the orthogonal projection of $H$ onto $H^N$ satisfying $\|P^N\| \leq 1$ and as $N \to \infty$ we have $\|P^N x - Px\| \to 0$ for all $x \in H$. The resulting approximating Riccati equation becomes

$$\mathcal{F}^N(X^N) = [A^N]^*X^N + X^N A^N - X^N B^N (B^N)^* X^N + (C^N)^* C^N = 0. \tag{6}$$

Let $X_k \in \mathcal{L}(H)$ denote the iterates of the Newton method for the infinite dimensional Riccati equation $\mathcal{F}(X) = 0$. Likewise, $X_k^N \in \mathcal{L}(H^N)$ denotes the iterates of the Newton method for the discretized Riccati equation $\mathcal{F}^N(X^N) = 0$.

We say that the approximation scheme *converges* if

$$\lim_{N \to +\infty} \|X_{opt}^N - P^N X_{opt}\| = 0. \tag{7}$$

We have shown in [9] that the Newton iterates $X_k$ converge strongly to $X_{opt}$ and the convergence is quadratic. Now assume that one applies a Newton type algorithm to (5) and (6) which produces quadratically convergent iterations $X_k$ and $X_k^N$, $k = 1, 2, \ldots$. For a given $\varepsilon > 0$, $X_0 \in D(\mathcal{F})$ and $X_0^N \in D(\mathcal{F}^N)$ define the numbers $M(\varepsilon, x_0)$ and $M^N(\varepsilon, x_0^N)$ by

$$M(\varepsilon, X_0) \triangleq \inf\{k : \|X_k - X_{opt}\| < \varepsilon\} \quad \text{and} \quad M^N(\varepsilon, X_0^N) \triangleq \inf\{k : \|X_k^N - X_{opt}^N\| < \varepsilon\},$$

respectively. Here, $X_0$ and $X_0^N$ are the starting values for the iterations. The Mesh Independence Principle (MIP) takes the form

$$M(\varepsilon, X_0) = M^N(\varepsilon, P^N x_0) + \tau(N), \tag{8}$$

where $\tau(N) \longrightarrow 0$ as $N \longrightarrow +\infty$. Also, assume there are constants $c$ and $c^N$ such that

$$\|X_{k+1} - X_{opt}\| \leq c\,\|X_k - X_{opt}\|^2 \tag{9}$$

and

$$\|x_{k+1}^N - x_\infty^N\| \leq c^N \left\|x_k^N - x_\infty^N\right\|^2, \tag{10}$$

respectively. Let $\hat{c}$ and $\hat{c}^N$ be the minimal values of $c$ and $c^N$ that satisfy (9) and (10) where $X_0^N = P^N X_0$. Since $P^N : H \longrightarrow H^N$ is the orthogonal projection of $H$ onto $H^N$, in some cases one can show that another form of the strong MIP is given by

$$\hat{c}^N = \hat{c} + \gamma(N), \tag{11}$$

where $\gamma(N) \longrightarrow 0$ as $N \longrightarrow +\infty$. The basic idea behind these strong versions of mesh independence is that the number of iterations required to achieve a given error tolerance is independent of the mesh size and asymptotically converges to the number of infinite dimensional iterations (theoretically) required to attain the same tolerance. We have established the following MIP (see [9] and [10]).

**Theorem**. Assume the approximation scheme is convergent, dually convergent and is uniformly stabilizable and uniformly detectable. If $B$ is compact, then there exist $\tilde{c}^N$ and $\tilde{c}$ such that

$$\|X_{k+1}^N - X_{opt}^N\| \leq \tilde{c}^N \|X_k^N - Xopt\|^2, \tag{12}$$

$$\|X_{k+1} - X_{opt}\| \leq \tilde{c}\|X_k - X_{opt}\|^2 \tag{13}$$

and $\tilde{c}^N = \tilde{c} + \delta(N)$, where $\delta(N) \longrightarrow 0$ as $N \longrightarrow +\infty$.

The MIP theorem above provides essential information about the construction of numerical methods for solving PDE Ricatti equations. For example, we applied a finite volume and conforming finite element scheme to a hybrid PDE system of hyperbolic type. Here $N$ is the dimension of the approximating subspace. The table below shows that the finite volume scheme is mesh independent, but the finite element scheme is not. For this problem the conforming finite volume scheme is **not** dual convergent.

| $N$ | $\hat{c}_{FV}^N$ | $\hat{M}_{FV}^N$ | $\hat{c}_{FE}^N$ | $\hat{M}_{FE}^N$ |
|---|---|---|---|---|
| 8 | $1.01497 \times 10^{-2}$ | 3 | $3.98797 \times 10^6$ | 14 |
| 16 | $6.12638 \times 10^{-3}$ | 3 | $3.31429 \times 10^6$ | 12 |
| 32 | $6.18756 \times 10^{-3}$ | 3 | $1.81757 \times 10^6$ | 10 |
| 64 | $6.57817 \times 10^{-3}$ | 3 | $5.04459 \times 10^5$ | 8 |
| 128 | $7.11713 \times 10^{-3}$ | 3 | $3.04030 \times 10^4$ | 7 |
| 256 | $7.78503 \times 10^{-3}$ | 3 | $7.82120 \times 10^4$ | 7 |
| 512 | $8.55008 \times 10^{-3}$ | 3 | $6.02653 \times 10^3$ | 6 |
| 1024 | $9.32057 \times 10^{-3}$ | 3 | $1.59829 \times 10^2$ | 5 |

Table 1: Convergence rates and iteration counts for a finite volume and finite element scheme.

This work is a preliminary step in the development of general computational tools for large scale Riccati equations that arise in a variety of control and estimation problems. The results imply that, even when the Riccati equations are used for applications such as weather prediction, control theory concepts provide the basic information about the type of approximation schemes that produce mesh independence.

Personnel Supported Under the Grant

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| Eugene M. Cliff | | Hoan Nguyen |
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| | | James Vance |
| | | Eric Vugrin |
| | | Adam Childers |
| | | Dan Sutton |

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## Honors and Awards

Professor John Burns was named Honorary Professor of Mathematics at Beijing Institute of Technology, November, 2005

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## Interactions and Transitions

Our efforts to expedite the transition of our research to industrial and Air Force needs were manifested by direct industrial/laboratory interactions and participation at professional meetings. One of the major components of this effort remains an active cooperation and coordination with the Air Force Research Laboratory (AFRL) and with our industrial partners.